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Embedding $U_q(sl(2))$ and Sine Algebras in Generalized Clifford Algebras

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We establish the connection between certain quantum algebras and generalized Clifford algebras (GCA). To be precise, we embed the quantum tori Lie algebra and $U_q(sl(2))$ in GCA.

1. INTRODUCTION

Here we establish a connection between certain algebraic structures and generalized Clifford algebras GCA [1–3]. We show that the quantum tori Lie algebra (QTLA), alias sine trigonometric or Fairlie-Fletcher-Zachos (FFZ) algebra [4, 5], can be constructed from GCA. Relying on the fact that $U_q(sl(2))$ can be constructed from QTLA [6, 7], we give the embedding of the quantum universal enveloping algebra $U_q(sl(2))$ in GCA.

To begin, we recall that the classical Clifford algebras have in common their definition from a quadratic or bilinear relation and consequently admit a $\mathbf{Z}_2$-graded structure. However, mathematicians have obtained, in the spirit of the usual Clifford algebras, new algebras defined from an *n*-linear relation and leading to an underlying $\mathbf{Z}_n$-graded structure, the so-called generalized Clifford algebras, which emerge naturally in various contexts [8–10]. The latter endow a differential structure on noncommutative variables which allows us to build a theory beyond supersymmetry [11, 12].

This paper is organized as follows: In Section 2 we sketch briefly the basic and useful properties of GCA. Then we construct the quantum tori Lie

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algebra in Section 3. In Section 4 we give the embedding of the quantum universal enveloping algebra $U_q(sl(2))$ in GCA.

2. REVIEW OF THE GCA

In this section, we recall briefly the basic notions connected with GCA (for more details see, e.g., refs. 2 and 3). The generalized Clifford algebra $\mathbf{C}_n^r$ is generated by a set of r canonical generators $\Gamma_1, \Gamma_2, \ldots, \Gamma_i$ fulfilling

$$\begin{aligned} \Gamma_i\Gamma_j &= \omega\Gamma_j\Gamma_i, \qquad i < j \\ \Gamma_i^n &= 1, \qquad i = 1, 2, \ldots, r \end{aligned} \tag{1}$$

where $\omega = \exp(2\pi i/n)$ is an nth primitive root of unity.

If we substitute the equation in the second line (i.e, $\Gamma_i^n = 1$) by $\Gamma_i^n = 0$, the obtained algebra becomes the generalized Grassmann algebra $\mathbf{G}(r, n)$. The latter is the fundamental tool in fractional statistics, fractional supersymmetry [11], and even 2D fractional conformal theory [12].

3. QUANTUM TORI LIE ALGEBRA AND ITS GCA REALIZATION

The quantum tori Lie algebra, also called trigonometric sine algebra or FFZ algebra, is generated by the elements J_m, where $m \equiv (m_1, m_2)$ is any vector belonging to the square integer lattice $Z^2 - \{0, 0\}$, with the commutation relations

$$[J_{(m_1,m_2)}, J_{(m_1',m_2')}] = -2i \sin\left(\frac{2\pi}{k}(m_1m_2' - m_1'm_2)\right)J_{(m_1+m_1',m_2+m_2')} \tag{2}$$

This is exactly the Moyal bracket quantization of the area-preserving diffeomorphism or symplectomorphism algebra on the 2D torus [13]:

$$L_{(m_1,m_2)} = -i\epsilon^{\alpha\beta}m_\alpha \exp i(m_1\sigma_1 + m_2\sigma_2)\partial_\beta \tag{3}$$

where $\partial_i = \partial/\partial\sigma_i$ and $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = 1$.

It should be mentioned that the deformation here is the Moyal quantization, which is strongly different from the Drinfel'd and Jimbo one where the Hopf structure plays a crucial role.

Another approach to the definition of QTLA is based on the idea of noncommutative geometry [13, 14].

Now we construct the QTLA from the GCA. We have the following assertion:

Theorem 1. The generators $T^{(i,j)}_{(m_1,m_2)}$, $i < j$, $(m_1, m_2) \neq (n, n)$, defined by

$$T^{(i,j)}_{(m_1,m_2)} = w^{(m_1,m_2/2)}\Gamma_i^{m_1}\Gamma_j^{m_2} \tag{4}$$

satisfying relation (2), determine the quantum tori Lie algebra through the identification $T^{(i,j)}_{(m_1,m_2)} \sim J_{(m_1,m_2)}$, and where we have used $n = k$.

The proof follows from the relation $\Gamma_i^m\Gamma_j^{m'} = w^{mm'}\Gamma_j^{m'}\Gamma_i^m$ after carrying out some algebraic manipulations.

4. THE EMBEDDING OF $U_q(sl(2))$ IN GCA

In this section we give the GCA realization of $U_q(sl(2))$; the latter emerges in several contexts, e.g., sine-Gordon theory [15] and Chern–Simons theory [6], which is connected with the quantum Hall system. The quantum universal enveloping algebra $U_q(sl(2))$ is defined as a complex unital associative algebra consisting of polynomials in $X^{\pm}$ and convergent power series in h so that ($q \neq 0, 1$)

$$[h, X^{\pm}] = \pm X^{\pm} \qquad \text{and} \qquad [X^+, X^-] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}} \tag{5}$$

The symbols $q^{\pm 2h}$ are usually considered as generators; including h in $U_q(sl(2))$ allows the limit $q \to 1$, which reduces Eq. (5) to the defining relation of the Lie algebra $sl(2)$. In what follows, we embed $U_q(sl(2))$ in GCA; we have the following theorem:

Theorem 2. The generators $X^{\pm}$ and $q^{\pm 2h}$ defined by

$$\begin{aligned} X^+ &= \frac{T^{(i,j)}_{(1,1)} - T^{(i,j)}_{(-1,1)}}{(q - q^{-1})} \\ X^- &= \frac{T^{(i,j)}_{(-1,-1)} - T^{(i,j)}_{(1,-1)}}{(q - q^{-1})} \\ q^{+2h} &= T^{(i,j)}_{(-2,0)} \\ q^{-2h} &= T^{(i,j)}_{(2,0)} \end{aligned} \tag{6}$$

where the deformation parameter is taken to be $w = q$, satisfy the commutation relation (2) determining the algebra $U_q(sl(2))$.

This embedding can be extended to additional cases. The following construction depends on the pair $m, m' \in Z^2$ and four complex parameters a, b, c, and d:

Theorem 3. The following generators satisfy the $U_q(sl(2))$ algebra

$$
\begin{aligned}
X^{+} &= \frac{aT^{(i,j)}_{(m_1,m_2)} + bT^{(i,j)}_{(m_1,m'_2)}}{(q - q^{-1})} \\
X^{-} &= \frac{cT^{(i,j)}_{(-m_1,-m_2)} + aT^{(i,j)}_{(-m'_1,-m'_2)}}{(q - q^{-1})} \\
q^{+2h} &= T^{(i,j)}_{(m_1-m'_1,m_2-m'_2)} \\
q^{-2h} &= T^{(i,j)}_{(m'_1-m_1,m'_2-m_2)}
\end{aligned}
\tag{7}
$$

where here the deformation parameter $q = w^{(m\times m')/2}$ and $(m \times m') = m_1m'_2 - m_2m'_1$.

Calculating the commutation relation for $X^{\pm}$ and $q^{\pm 2h}$ and using Eg. (1) to get the commutation relation for $U_q(sl(2))$ gives the choice $ad = bc = 1$.

5. CONCLUSION

We have established the connection between certain quantum algebras and the generalized Clifford algebras. In particular, we have embedded the quantum tori Lie algebra in GCA; based on this, we have proposed the embedding of $U_q(sl(2))$ in GCA.

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